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CLASSIFICATION OF PATTERNS IN THREE–QUBIT SYSTEM IN QUANTUM NEURAL NETWORK

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The study of the classification of patterns of two different categories has been undertaken in the framework of quantum neural network (QNN) in a three- qubit system using the method of repeated iterations in Grover's algorithm. Operator describing an inversion about average has been constructed as a square matrix of order eight, the phase inversion operators and corresponding iteration operators for patterns separately representing two different categories have been derived and various possible superposition as the choice for search states for the classification of these patterns have been obtained for starting states consisting of two patterns and a single pattern respectively. It has also been demonstrated that on second iteration of the exclusion superposition by the corresponding iteration operators, the patterns are most suitably classified using the Grover's algorithm with two - pattern start- system. Evaluating different probabilities for the separate classifications of patterns of two different categories, with one-pattern start-states respectively, on operating various possible superposition separately by the corresponding iterative operators, it has been demonstrated that the respective supervision of inclusion as the search states provide the ideal choices for the classification of these patterns.

INTRODUCTION

In last fifteen years quantum entanglement [1] has played important role in the fields of quantum information theory[2, 3], quantum computers[4], universal quantum computing network[5], teleportation[6], dense coding[7, 8], geometric quantum computation[9,10] and quantum cryptography[11-13]. Basis of entanglement is the correlation that can exist between qubits. From physical point of view, entanglement is still little understood. What makes it too powerful is the fact that since quantum states exist as superposition, these correlations exist in superposition as well and when superposition is destroyed, the proper correlation is somehow communicated between the qubits [14]. It is this communication that is the crux of entanglement. Entanglement is one of the key resources required for quantum computation and hence the experimental creation and measurement of entangled states is of crucial importance for various physical implementations of quantum computers.

Singh and Rajput have recently explored [15] the entanglement as one of the key resources required for quantum neural network (QNN), established [16] the functional dependence of the entanglement measures on spin correlation functions, worked out the correspondence between evolution of new maximally entangled states (Singh-Rajput MES) of two-qubit system and representation of SU(2) group, and investigated [16] the evolution of MES under a rotating magnetic field. They have also performed the pattern association (quantum associative memory) [17, 18] and pattern classifications [19, 20] by employing the method of Grover's iteration [21] on Bell's MES [22] and Singh-Rajput MES in two-qubit system and demonstrated that for all the related processes (memorization, recalling, and pattern classification) in a two-qubit system Singh-Rajput MES provide the most suitable choice of memory states and the search states. Applying the method of Grover's iterate on three different superposition in three-qubit system, it has been shown [20, 23] that the choice of exclusive superposition, as the search state, is most suitable one for the desired pattern classifications based on Grover's iterative search algorithm.

In the present paper the study of the classification of patterns P_1 and P_2 of two different categories has been undertaken in the framework of quantum neural network (QNN) in a three-qubit system using the method of repeated iterations in Grover's algorithm. Operator describing an inversion about average has been constructed as a square matrix of order eight, the phase inversion operators and corresponding iteration operators for patterns separately representing two different categories have been derived and various possible superposition as the choice for search states for the classification of these patterns have been obtained for starting states consisting of two patterns and a single pattern respectively. The probabilities of correct classification of the pattern P_2, incorrect classification of other patterns in class of pattern P_2 and irrelevant classification (other than that in which we are interested *i.e.* the patterns not belonging to class of P_2) respectively and the total probability of desired classifications and the conditional probability of correct classification of the pattern P_2 on repeated iterations of all possible superposition for two- pattern start-states have been evaluated and tabulated in three tables. Analyzing these results and also their graphical comparative behavior, it has been shown that the superposition of exclusion is the most suitable choice as the search state for classification of patterns P_2 by using the Grover's algorithm of repeated iterations with two-pattern start- state. All these probabilities have also been evaluated for the classification of patterns P_1 and it has been shown that on second iteration of the superposition of exclusion by the corresponding iteration operator the pattern P_1 is also most suitably classified using the Grover's algorithm with two- pattern start state. It has also been argued that on second iteration of the exclusion superposition by the corresponding iteration operators the patterns P_2 and P_1 respectively are most suitably classified using the Grover's algorithm with two -pattern start- system.

Different probabilities have been evaluated for the separate classifications of P_2 and P_1 with one-pattern start-states P_2 and P_1 respectively, on operating various possible superposition separately by the corresponding iterative operators and it has been demonstrated that the respective superposition of inclusion as the search states provide the ideal choices for the classification of patterns P_1 and P_2 respectively. It has also been demonstrated that on the repeated iterations by these iteration operators respectively, the respective superposition of exclusions are the most suitable choice as the respective search states for the separate classifications of patterns P_2 with patterns P_1 as one- pattern start -state and that of P_1 with Pattern P_2 as start-state.

STATE PREPARATION FOR GROVER'S ITERATIVE METHOD OF PATTERN CLASSIFICATION

Let us consider $B = \{0,1\}$ and let the set $T = \{(x_i, y_i)\}$ be a set of m pairs of points, x_i in B^n and y_i in B. In pattern classification a quantum system is constructed for correctly labeling points in T and for generalizing in a reasonable way to label other points belonging to B^n , which do not belong to T. Thus in pattern classification such a quantum system is constructed that approximates the function $f: B^n \rightarrow B$ from which the set T was drawn. There are three different approaches to state preparation, based on information in set *T* of (*n* + 1) two states quantum systems ($|0>$ and $|1>$);

(i) Inclusion, (ii) Exclusion, (iii) Phase Inversion

Inclusion is most intuitive where basis states not in *T* have zero coefficients and those in *T* have non-zero coefficients in the superposition:

$$|\Psi_{inc}> = \frac{1}{\sqrt{m}} \sum_{(x_i y_i \epsilon T} |x_i\, y_i> \quad \text{.... (2.1)}$$

Exclusion is an opposite approach, where basis states in *T* have zero coefficients and those not in *T* have non-zero coefficients in the superposition:

$$|\Psi_{exc}> = \frac{1}{\sqrt{2^n - m}} \sum_{(x_i y_i \in T} |x_i\, y_i> \quad \text{... (2.2)}$$

In Phase Inversion, all basis states are included with coefficients of equal amplitudes but with different phases based on membership in *T*:

$$|\Psi_{ph}> = \frac{1}{\sqrt{2^n}} (\sum_{x_i y_i \notin T} |x_i\, y_i> - \sum_{x_i y_i \epsilon T} |x_i\, y_i>) \quad \text{... (2.3)}$$

After state preparation, the pattern classification may be performed in straight forward approach employing the method of Grover's iterate which is described as a product of unitary operators $\hat{G}\hat{R}$ applied to quantum state iteratively and probability of desired result maximized by measuring the system after appropriate number of iterations. Here the operator $\hat{R}$ is phase inversion of the state(s) that we wish to observe upon measuring the system. It is represented by identity matrix I with diagonal elements corresponding to desired state(s) equal to -1 and the operator $\hat{G}$ descried as an inversion about average:

If $\left|\Psi> = \frac{1}{\sqrt{2^{n+1}}} \sum_{(x_i y_i) \in B^n} \right| x_i\, y_i >$ then $G = 2\left|\Psi><\Psi\right| - I$ … (2.4)

For a three qubit system the general data is represented as state of superposition

$$|\Psi \geq \frac{1}{\sqrt{8}} [\,|000> \;+|001> +|010> +|011> +|100> +|101> +|110> +|111>] \quad \ldots (2.5)$$

Substituting this superposition into eqn. (2.4), we get the following operator $\widehat{G}$, describing an inversion about average:

$$\hat{G} = \frac{1}{4}[g_{ij}]\,, \quad \ldots (2.6)$$

where $g_{ii} = -3$ *and* $g_{ij} = 1$ *for* $i \neq j$, with i, j = 1, 2 …… 8

Thus we have

$$\hat{G} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{bmatrix} \quad \ldots (2.7)$$

The phase inversion matrix $\hat{R}$ may be constructed according to the patterns to be classified and then multiplying this matrix with the matrix $\hat{G}$ of equation (2.7) we may get the following unitary matrix to be used in the method of Grover's iterate

$$\widehat{D} = \hat{G}\hat{R} \quad \ldots (2.8)$$

which can be applied onto quantum search state iteratively and probability of desired correct pattern classification can be maximized by measuring the system after appropriate number of iterations.

CLASSIFICATION OF PATTERNS OF TWO DIFFERENT CATEGORIES RESPECTIVELY

Method of Grover's iteration for pattern classification in a three qubit system can be applied for the separate classifications of patterns P_1 and P_2 by using Quantum Neural Network (QNN). The dealer wants a machine with a set of sensors, which measures three properties (parameters) of the objects of these categories : shape, texture and weigh:

$$P = \begin{bmatrix} shape \\ texture \\ weight \end{bmatrix} \quad \ldots (3.1)$$

The sensor with output as the shape will give 1 if the object is round and 0 if it is elliptical. The texture sensor will give the output 1 if it is smooth and 0 if it is rough and the weight sensor will give the output 1 if weight of the object is greater than 1 pound and 0 if weight is less than 1 pound. Therefore, a prototype of the objects of first category would be represented by the pattern

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } |100> \quad \ldots (3.2)$$

and a prototype of the object of second category would represented by the pattern

$$P_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or } |110> \quad \ldots (3.3)$$

Based on minimum Hamming distance the patterns |000>, |001>, |100> and |101> belong to the class C_1 containing |100> and other patterns of the usual three qubit system *i.e.* |010>, |011>, |110> and |111> belong to the class C_2 containing |110>. Let us find the respective probabilities of classifications of patterns P_2 and patterns P_1 separately in the following subsections.

(A) Classification of Patterns P_2

For the classification of pattern P_2, given by eqn. (3.3), the phase inversion operator $\hat{R}$ of eqn. (2.8) can be represented in terms of the following matrix

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (3.4)$$

Substituting it into eqn, (2.8), with operator $\hat{G}$ given by eqn. (2.7), we get the following operator of the method of Grover's iterations for classification of the state |110> (P_2) :

$$\hat{D} = \hat{G}\hat{R} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -3 \end{bmatrix} \quad \ldots (3.5)$$

Let us consider the following choices of start states for the classification of the pattern P_2.

(i) Two-Patterns State

Let us consider the start state as consisting of the patterns P_1 and P_2. Then the set T of Section-2 is given by

$$T = \{(x_i, y_i)\} = \{100, 110\} \quad \ldots (3.6)$$

Then superposition, given by equation (2.1)-(2.3), may be written as

$$|\Psi inc >= \frac{1}{\sqrt{2}}[|100 > +|110 >] ,\ldots \quad (3.7)$$

$$|\Psi exc >= \frac{1}{\sqrt{6}}[|000 > +|001 > +|010 > +|011 > +|101 > +|111 >] \quad \ldots (3.8)$$

and $|\Psi phi >= \frac{1}{\sqrt{8}}[|000 > +|001 > +|010 > +|011 > -|100 > +|101 > -|110 > +|111 >]$ … (3.9)

or $$\Psi inc >= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\0\\1\\0\\1\\0\end{bmatrix} \quad |\Psi exc >= \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\1\\1\\0\\1\\0\\1\end{bmatrix} \quad |\Psi \mathrm{ph}i >= \frac{1}{\sqrt{8}}\begin{bmatrix}1\\1\\1\\1\\-1\\1\\-1\\1\end{bmatrix} \quad \ldots (3.10)$$

where '1' denotes the presence of the corresponding Eigen state in the superposition and '0' denotes its absence

With this operator let us find the probability of correct classification upon measurement on each iteration of Grover's search application on all three superposition $|\Psi inc >, |\Psi exc >$ $and\ |\Psi phi >$. For the classification of P_2 *l*et us denote the probabilities of correct classification (110), incorrect classification (other patterns in class C_2 i.e. 010,011 and111) and irrelevant classification (other than that in which we are interested i.e. patterns not belonging to class C_2) as P_C , P_W and P_R respectively. The total probability of desired pattern classification (patterns of class C_2) is

$$P = P_C + P_W \quad \ldots (3.11)$$

and the conditional probability (if the desired pattern is classified then the probability that the classification will be the correct one) may be written as

$$P_{cond} = \frac{P_C}{P_C + P_W} \quad \ldots (3.12)$$

Let us find all these probabilities after different iterations of the operator $\widehat{D}$ of eqn. (3.5) by using $|\Psi inc >, |\Psi exc > and\ |\Psi phi >$. as respective search states. After its first iteration on $|\Psi inc >$ the various probabilities are obtained as:

$$P_C = 0.5; P_W = 0; P_R = 0.5; P = 0.5 \text{ and } P_{cond} = 1 \qquad \dots (3.13)$$

These probabilities after different number of iterations of the operator $\widehat{D}$ are given in the Table- 3.1 where the first row shows the number of iterations and all these probabilities have been shown in other respective rows after various iterations of Grover's search algorithm on $|\Psi inc>$.

TABLE 3.1: Classification of P_2 taking both |100> and |110> as start state with $|\Psi inc>$ Superposition

PATTERN	RES1	RES2	RES3	RES4	RES5	RES6	RES7	RES8	RES9
Pc	0.49999	0.12503	0.03126	0.38279	0.56445	0.25827	0.00012	0.24177	0.56025
Pw	0	0.3751	0.09377	0.21099	0.05275	0.03663	0.44861	0.04037	0.05956
Pr	0.49999	0.50013	0.87505	0.40627	0.3828	0.7051	0.55127	0.71786	0.38014
P	0.49999	0.50013	0.12503	0.59378	0.6172	0.2949	0.44873	0.28214	0.61981
Pcond	1	0.25	0.25	0.64466	0.91454	0.87579	0.00027	0.85692	0.90391
Probabilities after various iterations of Grover's Search Algorithm on |Ψinc>									

This table shows that with the supervision $|\Psi inc>$ as the search state the probability of the correct classification of pattern P_2 never exceeds 56% up to nine iterations while the probability of classification of irrelevant patterns becomes as high as 87.5% on third iteration and it never falls below 38%. Thus the superposition $|\Psi inc>$ is not a suitable choice of search state for the classification of pattern P_2 by using Grover's method of repeated iterations.

After first iteration of the operator $\widehat{D}$ of eqn. (3.5) on $|\Psi exc>$, we get the various probabilities as

$$P_C = 0.375034; P_W = 0.12497; P_R = 0.500004; P = 0.875054 \text{ and } P_{cond} = 0.857183 \qquad \dots (3.14)$$

These probabilities after different number of iterations of the operator $\widehat{D}$ on $|\Psi exc>$ are given in the following table (Table-3.2).

TABLE 3.2: Classification of patterns P_2 with Superposition $|\Psi exc>$ as search state

PATTERN	RES1	RES2	RES3	RES4	RES5	RES6	RES7	RES8	RES9
Pc	0.37503	0.84383	0.58599	0.05272	0.17724	0.74149	0.75812	0.19793601	0.04133
Pw	0.12497	0.03127	0.19538	0.23638	0.4107	0.01478	0.01102	0.40011312	0.24044
Pr	0.5	0.12503	0.21882	0.71096	0.41217	0.24365	0.23079	0.40197073	0.71833
P	0.5	0.8751	0.78137	0.28909	0.58794	0.75628	0.76914	0.59804913	0.28177
Pcond	0.75006	0.96426	0.74995	0.18235	0.30146	0.98045	0.98568	0.330969481	0.14668
Probabilities after various iterations of Grover's Search Algorithm on Algorithm on |Ψexc>									

This table shows that the probability of correct classification of the pattern P_2 (Apples) with the superposition $|\Psi phi>$ never exceeds 56.4 % (on third and seventh iteration) while the probability of irrelevant classification is as high as 87.5% (first and ninth iterations) and it never falls below 38%. The probability of desired classification also never exceed 60% in this case. Thus the superposion $|\Psi\text{phi}>$ is not a suitable choice of search state for the classification of pattern P_2 by using Grover's method of repeated iterations.

The comparative periodic behavior of probabilities of correct classification of pattern P_2 for all the three superposition $|\Psi inc>$, $|\Psi exc>$ and $|\Psi\text{phi}>$ respectively as the search state in the method of Grover's repeated iterations is being shown in Fig-3.1. It is observed that the probability of correct classification never exceed the limit of 56% in case of inclusion $|\Psi_{inc}>$ and Phase inversion $|\Psi_{phi}>$, whereas in the case of exclusion this probability reaches up to 84% in second iteration. Thus the choice of inclusion superposition $|\Psi_{inc}>$ and phase inversion superposition $|\Psi_{phi}>$ as the search state for the desired pattern classification based on Grover's iterative search algorithm are not suitable, whereas the choice of $|\Psi_{exc}>$ as search state for the desired pattern classification based on Grover's iterative search algorithm is most suitable if the measurement is made after second iteration.

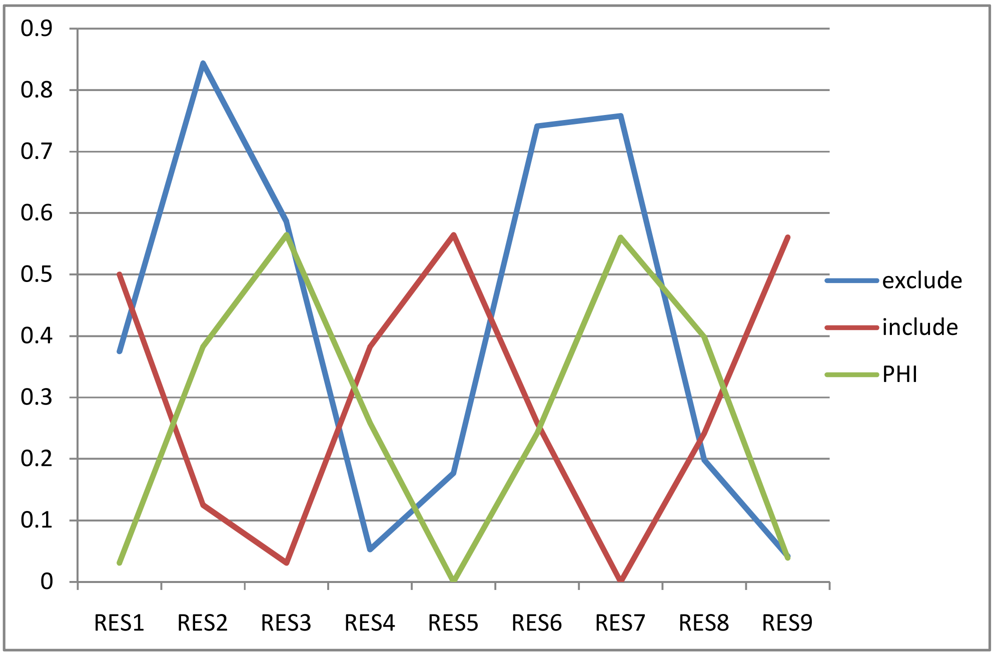

Figure-3.1: Probability of Correct Classification of Patterns P_2 for Three Different Superposition

The similar comparative periodic behavior of total probability of desired classification of pattern P_2 for all the three superposition $|\Psi inc>$, $|\Psi exc>$ and $|\Psi\text{phi}>$ respectively as the search state in the method of Grover's repeated iterations is shown in Fig-3.2 which also demonstrate the suitability of the superposition $|\Psi exc >$ as the choice for proper search state for the classification of the pattern P_2 in the method of Grover's algorithm of repeated iterations.

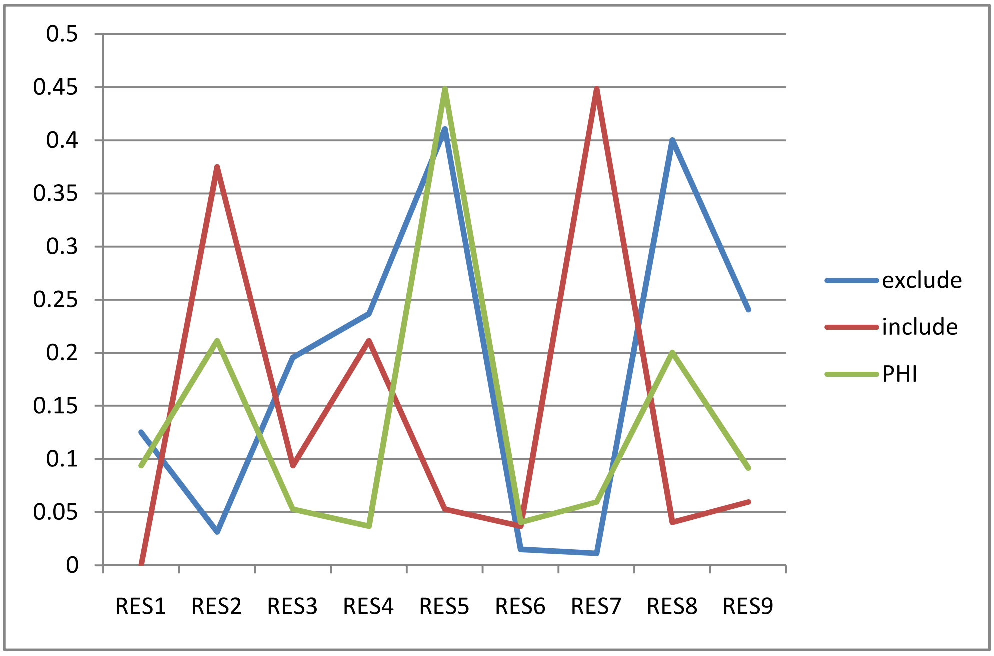

Figure-3.2: Total Probability of Desired Classification of P_2 for Three Different Superposition

The comparative periodic behavior of probability of irrelevant classifications of pattern P_2 for all the three superposition $|\Psi inc >$, $|\Psi exc >$ and $|\Psi \text{phi} >$ respectively as the search state in the method of Grover's repeated iterations is shown in Fig-3.3. It shows that the probability P_R of the irrelevant classifications is as high as 87% on first and ninth iterations if the superposition $|\Psi \text{phi} >$ is chosen as the search state in the method of Grover's repeated iterations for pattern classification. In the case of superposition $|\Psi inc >$ this probability of irrelevant classifications is more than 87.5% on third and eleventh iterations. In view of such high possibilities of irrelevant classifications none of these supervisions, $|\Psi \text{phi} >$ or $|\Psi inc >$ is suitable as the search state for the requisite patter classification. On the other hand, when the superposition $|\Psi exc >$ is chosen as the search state in the process of pattern classification based on Grover's algorithm, the probability of irrelevant classifications never exceeds 71% and it is lowest (negligibly small) on second iteration (when the probability of correct pattern classification is maximum). Thus the superposition $|\Psi exc >$ is the most suitable choice as a proper search state for the correct classification of the pattern P_2 in the method of repeated iterations based on Grover's algorithm.

Figure-3.4 shows the comparative behavior of conditional probability (if the desired pattern is classified then the probability that the classification will be the correct one) P_{cond} given by eqn. (3.12). This figure also demonstrate the superiority of the superposition $|\Psi exc>$ as the choice of search state in the classification of pattern P_2 if the measurement (classification) is made on second iteration, since the conditional probability shown in this figure is maximum on second iteration for this superposition $|\Psi exc>$ in comparison to the probabilities for other superposition $|\Psi inc>$ or $|\Psi \text{phi}>$.

Figure-3.3: Probability of Irrelevant Classification for Three Different Superposition

Figure-3.4 shows the comparative behavior of conditional probability (if the desired pattern is classified then the probability that the classification will be the correct one) P_{cond} given by eqn. (3.12). This figure also demonstrate the superiority of the superposition $|\Psi exc>$ as the choice of search state in the classification of pattern P_2 if the measurement (classification) is made on second iteration, since the conditional probability shown in this figure is maximum on second iteration for this superposition $|\Psi exc>$ in comparison to the probabilities for other superposition $|\Psi inc>$ or $|\Psi \text{phi}>$.

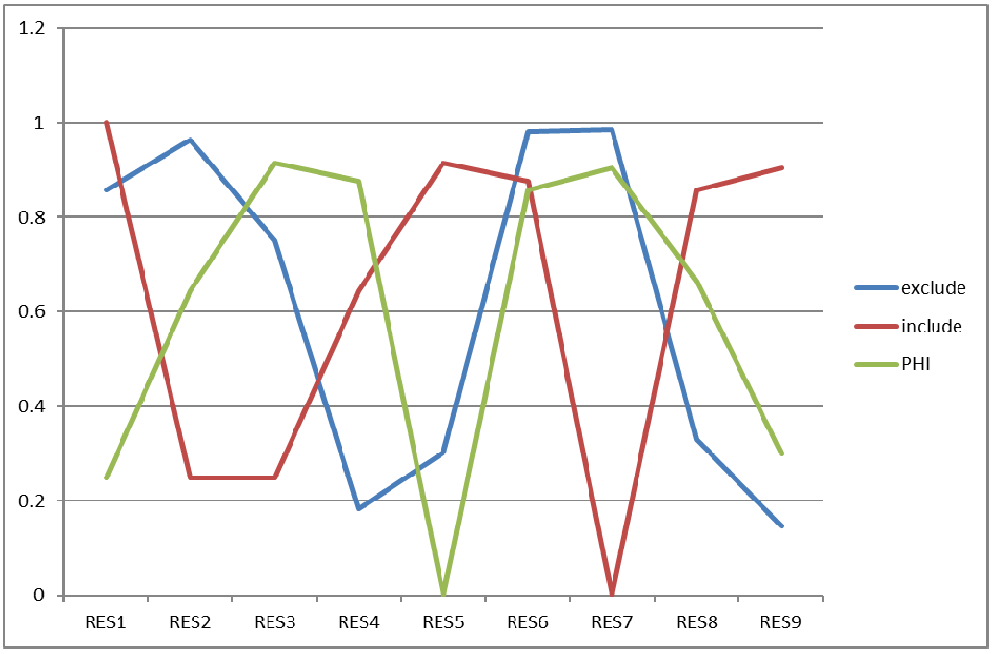

Figure-3.4: Conditional Probability of Correct Classification for Different Superposition

(ii) One Pattern State

Let us first take pattern P_1 as the start state for the classification of pattern P_2 , given as |110>, by the method of Grover's algorithm based on repeated iterations of the operator $\widehat{D}$ given by eqn.(3.5). Thus we have $T = \{(x_i, y_i)\}= \{|100 >\}$. Then using eqns. (2.1), (2.2) and (2.3) we get the following different superposition as the possible choices for search state.

$$|\Psi inc > \ = [|100 >] \quad \dots (3.15)$$

$$|\Psi exc > = \frac{1}{\sqrt{7}}[|000 > +|001 > +|010 > +|011 > +|101 > +|110 > +|111 >] \dots (3.16)$$

$$|\Psi phi > = \frac{1}{\sqrt{8}}[|000 > +|001 > +|010 > +|011 > -|100 > +|101 > +|110 > +|111 >] \dots (3.17)$$

which may also be written as

$$|\Psi inc > \ = \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} \qquad |\Psi exc >= \begin{bmatrix}1\\1\\1\\1\\0\\1\\1\\1\end{bmatrix} \qquad |\Psi \text{phi} >= \begin{bmatrix}1\\1\\1\\1\\-1\\1\\1\\1\end{bmatrix} \qquad (3.18)$$

The first three repeated operations of the operator $\widehat{D}$, given by eqn.(3.5), on the superposition $|\Psi inc>$ respectively give the following probabilities of correct pattern classification (P_C); incorrect classification (P_W); irrelevant classification (P_R), desired classification (P); and the conditional classification (P_{cond}):

$$P_C = 6.25\%;\ P_W = 18.75\%;\ P_R = 75\%;\ P = 25\%;\ P_{cond} = 25\% \quad \ldots (3.19)$$

$$P_C = 14.06\%;\ P_W = 4.69\%;\ P_R = 81.25\%;\ P = 18.75\%;\ P_{cond} = 77.87\% \quad \ldots (3.20)$$

$$P_C = 14.36\%;\ P_W = 3.17\%;\ P_R = 82.5\%;\ P = 17.47\%;\ P_{cond} = 82.20\% \quad \ldots (3.21)$$

and on the fourth iteration the results of first iteration are repeated. Thus the probability of correct pattern classification with $|\Psi inc>$ as search state never exceeds the limit of 15% and the probability of irrelevant classification does not fall below 75% and hence this supervision is not a suitable choice as search state for the classification of pattern P_2 with pattern P_1 as start state in the method of pattern classification based on Grover's algorithm.

The results of various iterations on the superposition $|\Psi phi>$ are given in Table-3.3

TABLE 3.3: Classification of Apple taking |100> as start state with the Phase Inverse Superposition

PATTERN	RES1	RES2	RES3	RES4	RES5	RES6	RES7	RES8	RES9	RES10	RES11
Pc	0.49999	0.49999	0.12503	0.03126	0.38279	0.56445	0.25827	0.57699	0.24177	0.56025	0.39829
Pw	0.00000	0.00000	0.37510	0.09377	0.21099	0.05275	0.03663	0.18125	0.04037	0.05956	0.20016
Pr	0.49999	0.49999	0.50013	0.87505	0.40627	0.38280	0.70510	0.24167	0.71786	0.38014	0.40158
P	0.49999	0.49999	0.50013	0.12503	0.59378	0.61720	0.29490	0.75825	0.28214	0.61981	0.59844
Pcond	1.00000	1.00000	0.25000	0.25000	0.64466	0.91454	0.87579	0.76096	0.85692	0.90391	0.66554
Probabilities after various iterations of Grover's Search Algorithm on |Ψphi>											

It demonstrate the unsuitability of the superposition $|\Psi phi>$ as the search state for the classification of pattern P_2 with pattern P_1 as start state in the method of pattern classification based on Grover's algorithm.

In the similar manner the different probabilities of classification of pattern P_2 on various iterations of the superposition $|\Psi exc>$., taking **|100>** as start state have been calculated and it has been shown that with this superposition the probability of correct classification becomes higher than 87% on sixth iteration with the probability of irrelevant classification lower than 11% and the probability of desired pattern higher than 89% and the conditional probability of the correct pattern classification as high as 98.5%. Thus the superposition $|\Psi exc>$ is most suitable choice as search state here also.

Let us now take pattern P_2 as the start state for the classification of pattern P_2 , given as |110>, by the method of Grover's algorithm based on repeated iterations of the operator $\widehat{D}$ given by eqn.(3.5). Then using eqns. (2.1), (2.2) and (2.3) we get the following different superposition as the possible choices for search state.

$$|\Psi inc> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |\Psi exc>= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad |\Psi \text{ph}i>= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \qquad \ldots (3.22)$$

The first four repeated operations of the operator $\widehat{D}$, given by eqn. (3.5), on the superposition $|\Psi inc >$ of eqns. (3.22) respectively give the following probabilities:

$$P_C = 56.25\%;\ P_W = 18.75\%;\ P_R = 25\%;\ P = 75\%;\ P_{cond} = 75\% \qquad \ldots (3.23)$$

$$P_C = 67.5\%;\ P_W = 5.5\%;\ P_R = 27\%;\ P = 73\%;\ P_{cond} = 92\% \qquad \ldots (3.24)$$

$$P_C = 91.8\%;\ P_W = 6\%;\ P_R = 2.2\%;\ P = 97.8\%;\ P_{cond} = 93.9\% \qquad \ldots. (3.25)$$

$$P_C = 57\%;\ P_W = 13\%;\ P_R = 30\%;\ P = 70\%;\ P_{cond} = 81.4\% \qquad \ldots (3.26)$$

These results show that on third iteration of superposition $|\Psi inc >$ the probability of correct pattern classification is as high as 91.8%, the probability of irrelevant classification is negligibly small and the probability of the desired classification is as high as 97.8%. It is also interesting to note that with this superposition in this case the probability of correct classification never falls below 56% and the probability of irrelevant classification never exceeds beyond 30%. Thus this superposition as the search state provides the ideal choice for the classification of the pattern P_2 with its own starting state.

The repeated operations of the operator $\widehat{D}$, given by eqn. (3.5), on the superposition $|\Psi exc >$ and $|\Psi phi >$ of eqns (3.22) demonstrate here that the probability of the correct pattern classification does not exceed the limit of 10% and the probability of the irrelevant classification does not fall below 65% on any number of iterations with any of these superposition as the search state. Thus none of these superposition is the suitable choice for the classification of pattern P_2 with its own starting state in the Grover's algorithm of repeated iterations.

(B) Classification of Pattern P_1

For the classification of pattern P_1, given by eqn. (3.2), the phase inversion operator $\hat{R}$ of eqn. (2.8) can be represented in terms of the following matrix

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (3.27)$$

Substituting it into eqn, (2.8), with operator $\widehat{G}$ given by eqn. (2.7), we get the following operator $\widehat{D}$ of the method of Grover's iterations for classification of the state |100>:

$$\widehat{D} = \widehat{G}\widehat{R} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -3 \end{bmatrix} \quad \ldots(3.28)$$

First iterations of this operator on the search state $|\Psi inc >$ of eqns (3.10) with the start state of two patterns , given by eqn. (3.6), we get the following values of the different probabilities related with the classification of of pattern P_1;

$$P_C = 0.5;\ P_W = 0;\ P_R = 0.5;\ P = 0.5\ and\ P_{cond} = 1$$

which are exactly same values as given by eqn. (3.13) and the results of other iterations are the same as given in table-3.1. Thus for the classification of pattern P_1 also the superposition $|\Psi inc >$ is not a suitable choice of search state in Grover's method of repeated iterations.

In the similar manner, the repeated iterations of the operator $\widehat{D}$, given by eqn. (3.28), on the superposition $|\Psi exc >$ and $|\Psi phi >$ give the similar values of various probabilities as shown in Table-3.2 and Table-3.3 respectively and hence the superposition of exclusion is the most suitable choice as the search state for classification of P_1also by using the Grover's algorithm of repeated iterations. *Thus on second iteration of the superposition $|\Psi exc >$ of eqns. (3.10) by $\widehat{D}$ operators separately given by eqns. (3.5) and (3.28) the patterns P_2 and P_1 are most suitably classified respectively using the Grover's algorithm with two pattern start system given by eqn.(3.6).*

Let us now take pattern P_2 (|110>) as the single pattern start state for the classification of pattern P_1 by the method of Grover's algorithm based on repeated iterations of the operator $\widehat{D}$ given by eqn.(3.28). Then the three superposition defined by eqns. (2.2), (2.3) and (2.4) have the forms given by eqns. (3.22). The repeated operations of the operator $\widehat{D}$, given by eqn. (3.28), on the superposition $|\Psi inc >$ of eqns. (3.22) give the same results as shown by eqns. (3.19), (3.20) and (3.21). Thus the probability of correct pattern classification with $|\Psi inc >$ as search state never exceeds the limit of 15% and the probability of irrelevant classification does not fall below 75% and hence this superposition is not a suitable choice as search state for the classification of pattern P_1 with pattern P_2 as start state in the method of pattern classification based on Grover's algorithm.

Applying the operator $\widehat{D}$ of eqn.(3.28). on the superposition $|\Psi phi >$ of eqns. (3.22), we get

$$\widehat{D}\ |\Psi phi>=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\0\\1\\0\\1\\0\end{bmatrix} \qquad \dots (3.29)$$

which give the following probabilities of pattern classifications:

$$P_C = 0.5;\ P_W = 0;\ P_R = 0.5; P = 0.5: P_{cond.} = 1 \qquad \dots (3.30)$$

Further iterations of this superposition ($|\Psi phi>$) by operator $\widehat{D}$ given by eqn.(3.28) give the same results as shown in Table-3.3 . Thus the probability of correct pattern classifications does not exceed the limit of 50% in first five iterations and remains below 60% on any number of further iterations. It demonstrates the unsuitability of the superposition $|\Psi phi>$ as the search state for the classification of pattern P_1 with pattern P_2 as start state in the method of pattern classification based on Grover's algorithm.

In the similar manner the different number of iterations of the superposition $|\Psi exc>$ by operator $\widehat{D}$ given by eqn.(3.28) we get the same values demonstrating that with this superposition the probability of correct classification becomes higher than 87% on sixth iteration with the probability of irrelevant classification lower than 11% and the probability of desired pattern becomes higher than 89% and the conditional probability of the correct pattern classification as high as 98.5%. Thus the superposition $|\Psi exc>$ is most suitable choice as search state for the classification of pattern P_1 with pattern P_2 as start state in the method of repeated iterations based on Grover's algorithm.

Finally, let us take pattern P_1 as the start state for the classification of pattern P_1 by the method of Grover's algorithm based on repeated iterations of the operator $\widehat{D}$ given by eqn.(3.28). Different possible choices of search states are then given by eqns. (3.18). The first iteration of the superposition $|\Psi inc>$ of eqns. (3.18), we get

$$\widehat{D}|\Psi inc>=-\frac{1}{4}\begin{bmatrix}1\\1\\1\\1\\-3\\1\\1\\1\end{bmatrix} \qquad \dots (3.31)$$

which give following probabilities:

$$P_C = 0.5625;\ P_W = 0.1875;\ P_R = 0.25; P = 0.75: P_{cond.} = 0.75$$

which are the same results as given eqn. (3.23). Next three iterations of $|\Psi inc>$ of eqns. (3.18) by operator $\widehat{D}$ of eqn. (3.28) give the same values of all these probabilities as given by eqns. (3.24), (3.25) and (3,26). These results show that on third iteration of superposition $|\Psi inc >$ the probability of correct pattern classification is as high as 91.8%, the probability of irrelevant classification is negligibly small and the probability of the desired classification is as high as 97.8% and this superposition as the search state provides the ideal choice for the classification of the pattern P_1 with its own starting state.

The repeated operations of the operator $\hat{D}$ of eqn.(3.28), on the superposition $|\Psi exc >$ and $|\Psi phi >$ of eqns. (3.18) demonstrate here also that the probability of the correct pattern classification does not exceed the limit of 10% and the probability of the irrelevant classification does not fall below 65% on any number of iterations with any of these superposition as the search state and hence none of these superposition is the suitable choice as the search state for the classification of pattern P_1 with its own starting state.

DISCUSSION

For the classification of pattern-P_2 of eqn. (3.3) by using quantum neural network (QNN) in a three- qubit system, the phase inversion operator $\hat{R}$ has been obtained in the form of matrix given by eqn. (3.4) which leads to the operator $\hat{D}$, as given by eqn.(3.5),for repeated iterations in Grover's algorithm of pattern classification. Equations (3.10) give the possible superposition for the choice of search state with the two-pattern start state shown by eqn. (3.6). The probabilities P_C, P_W and P_R of correct classification of the patterns P_2, incorrect classification of other patterns in class C_2 i.e. 010,011 and111 and irrelevant classification (other than that in which we are interested i.e. patterns not belonging to class C_2) respectively and the total probability of desired classifications and the conditional probability of correct classification of the pattern P_2 on repeated iterations of the superposition $|\Psi inc >$ by the operator $\hat{D}$ of eqn. (3.5) are given in Table-3.1 showing that this superposition is not a suitable choice of search state for the classification of pattern P_2 by using Grover's method of repeated iterations. Table-3.3 of various probabilities of classifications of pattern P_2 on various iterations of the operator $\hat{D}$ of eqn. (3.5) on the superposition $|\Psi phi >$ shows that this superposition also is not a suitable choice of search state for the classification of pattern P_2 by using Grover's method of repeated iterations. Table-3.2 of all these probabilities after different number of iterations of the operator $\hat{D}$ on the superposition $|\Psi exc >$ shows with this superposition as search state we get suitably high probability of the correct classification of the pattern P_2 and hence the superposition of exclusion is the most suitable choice as the search state for classification of these pattern (P_2) by using the Grover's algorithm of repeated iterations with two-pattern start state given by eqn. (3.6).

Figure-3.1, describing the comparative periodic behavior of probabilities of correct classification of pattern P_2 for all the three superposition $|\Psi inc >$, $|\Psi exc >$ and $|\Psi \text{phi} >$ respectively as the search state in the method of Grover's repeated iterations, shows that the probability of correct classification never exceed the limit of 56% in case of inclusion $|\Psi_{inc}>$ and Phase inversion $|\Psi_{phi}>$, whereas in the case of exclusion this probability reaches up to 84% in second iteration. Thus the choice of inclusion superposition $|\Psi_{inc}>$ and phase inversion superposition $|\Psi_{phi}>$ as the search state for the desired pattern classification based on Grover's iterative search algorithm are not suitable, whereas the choice of $|\Psi_{exc}>$ as search state for the desired pattern classification based on Grover's iterative search algorithm is most suitable if the measurement is made after second iteration. Figure-3.2 showing the comparative periodic behavior of total probability of desired classification of pattern P_2 for all the three superposition $|\Psi inc>$, $|\Psi exc>$ and $|\Psi \text{phi}>$ respectively as the search state in the method of Grover's repeated iterations also demonstrates the suitability of the superposition $|\Psi exc>$ as the choice for proper search state for the classification of the pattern P_2 in the method of Grover's algorithm of repeated iterations. Fig-3.3 shows that the probability P_R of the irrelevant classifications is as high as 87% on first and ninth iterations if the superposition $|\Psi \text{phi}>$ is chosen as the search state in the method of Grover's repeated iterations for pattern classification. In the case of superposition $|\Psi inc>$ this probability of irrelevant classifications is more than 87.5% on third and eleventh iterations. In view of such high possibilities of irrelevant classifications also none of these supervisions, $|\Psi \text{phi}>$ or $|\Psi inc>$, is suitable as the search state for the requisite pattern classification. This figure also shows that when the superposition $|\Psi exc>$ is chosen as the search state in the process of pattern classification based on Grover's algorithm, the probability of irrelevant classifications never exceeds 71% and it is lowest (negligibly small) on second iteration (when the probability of correct pattern classification is maximum). Thus for the measurement on second iteration the superposition $|\Psi exc>$ is the most suitable choice as a proper search state for the correct classification of the pattern P_2 by the method of repeated iterations based on Grover's algorithm. Figure-3.4 showing the comparative behavior of conditional probability P_{cond} defined by eqn. (3.12), also demonstrates the superiority of the superposition $|\Psi exc>$ as the choice of search state in the classification of pattern P_2 if the measurement (classification) is made on second iteration, since the conditional probability shown in this figure is maximum on second iteration for this superposition $|\Psi exc>$ in comparison to the probabilities for other superposition $|\Psi inc>$ or $|\Psi \text{phi}>$.

Equations (3.18) give the possible superposition as the choice of search state taking pattern P_1 as single pattern start state for the classification of pattern P_2 by the method of Grover's algorithm based on repeated iterations of the operator $\widehat{D}$ given by eqn.(3.5). Relations (3.15)-(3.19) demonstrate that the probability of correct pattern classification with $|\Psi inc>$ as search state never exceeds the limit of 15% and the probability of irrelevant classification does not fall below 75% and hence this superposition is not a suitable choice as search state for the classification of patterns P_2 with pattern P_1 as start state. The results of various iterations of this operator on the superposition $|\Psi phi>$ are given in Table-3.3 demonstrating the unsuitability of the superposition $|\Psi phi>$ as the search state for the classification of pattern P_2 (Apples) with pattern P_1 as start state in the method of pattern classification based on Grover's algorithm. It has also been shown that by the operations of the operator $\hat{D}$, given by eqn.(3.5), on the superposition $|\Psi exc>$ the probability of correct classification becomes higher than 87% on sixth iteration with the probability of irrelevant classification lower than 11% and the probability of desired pattern higher than 89% and the conditional probability of the correct pattern classification as high as 98.5%. Thus the superposition $|\Psi exc>$ is most suitable choice as search state in this case also.

Equations (3.22) give different superposition as the possible choices for search state for the classification of pattern P_2 with the single pattern starting state as P_2 itself. Results (3.23)-(3.26) show that on third iteration of superposition $|\Psi inc>$ of eqns. (3.22) the probability of correct pattern classification is as high as 91.8%, the probability of irrelevant classification is negligibly small and the probability of the desired classification is as high as 97.8%. It is also interesting to note that with this superposition in this case the probability of correct classification never falls below 56% and the probability of irrelevant classification never exceeds beyond 30%. Thus this superposition as the search state provides the ideal choice for the classification of the pattern P_2 with its own starting state. On the other hand the repeated operations of the operator $\hat{D}$, given by eqn.(3.5), on the superposition $|\Psi exc>$ and $|\Psi phi>$ of eqns (3.22) demonstrate here that the probability of the correct pattern classification does not exceed the limit of 10% and the probability of the irrelevant classification does not fall below 65% on any number of iterations with any of these superposition as the search state. Thus none of these superposition is the suitable choice for the classification of pattern P_2 with its own starting state in the Grover's algorithm of repeated iterations.

Phase inversion operator $\hat{R}$ given by eqn. (3.27) for the classification of pattern P_1 leads to the operator $\hat{D}$, given by eqn. (3.28), for repeated iteration in Grover's algorithm of pattern classification. The results of its repeated iterations on the search state $|\Psi inc>$ of eqns. (3.10) with the start state of two patterns, given by eqn. (3.6), are the same as given in Table-3.1 for the classification of pattern P_2 with the same search state. In the similar manner, the repeated iterations of the operator $\hat{D}$, given by eqn. (3.28), on the superposition $|\Psi exc>$ and $|\Psi phi>$ give the similar values of various probabilities as shown in Table-3.2 demonstrating that on second iteration of the superposition $|\Psi \mathrm{exc}>$ of eqns. (3.10) by $\hat{\mathrm{D}}$ operators given by eqn. (3.28) the pattern P_1 is most suitably classified using the Grover's algorithm with two pattern start system given by eqn.(3.6).

The repeated operations of the operator $\hat{D}$, given by eqn. (3.28), on the superposition $|\Psi inc>$ of eqns. (3.22) give the same results as shown by eqns. (3.19), (3.20) and (3.21). Thus the probability of correct pattern classification with $|\Psi inc>$ as search state never exceeds the limit of 15% and the probability of irrelevant classification does not fall below 75% and hence this superposition is not a suitable choice as search state for the classification of pattern P_1 with pattern P_2 as start state in the method of pattern classification based on

Grover's algorithm. Similarly different iterations of the superposition ($|\Psi phi >$) of eqns. (3.22) (by operator $\widehat{D}$ given by eqn.(3.28) give the same results as shown in Table-3.3 demonstrating the unsuitability of the superposition $|\Psi phi >$ as the search state for the classification of pattern P_1 with pattern P_2 as start state in the method of pattern classification based on Grover's algorithm. In the similar manner the different number of iterations of the superposition $|\Psi exc >$ of eqns, (3.22) by operator $\widehat{D}$ given by eqn.(3.28) we get the same values of respective probabilities as for pattern P_2 with Pattern P_1 as the start state, demonstrating that the superposition $|\Psi exc >$ is most suitable choice as search state for the classification of pattern P_1 with pattern P_2 as start state in the method of repeated iterations based on Grover's algorithm.

Repeated iterations of $|\Psi inc >$ of eqns. (3.18) by operator $\widehat{D}$ of eqn. (3.28) give the same values of all these probabilities as given by eqns. (3.24), (3.25) and (3,26) showing that this superposition as the search state provides the ideal choice for the classification of the pattern P_1 with its own starting state. In the similar manner the repeated operations of the operator $\widehat{D}$ of eqn.(3.28) on the superposition $|\Psi exc >$ and $|\Psi phi >$ of eqns. (3.18) demonstrate here also that the none of these superposition is the suitable choice as the search state for the classification of pattern P_1 with its own starting state.

From the foregoing analysis it may be concluded that on second iteration of the superposition $|\Psi \text{exc} >$ of eqns. (3.10) by $\widehat{D}$ operator separately given by eqns. (3.5) and (3.28) the patterns P_2 and P_1 are most suitably classified respectively using the Grover's algorithm with two pattern start system given by eqn. (3.6). It may also be concluded that on the repeated iterations by these operators respectively, the superposition $|\Psi exc >$ given by eqns. (3.28) and (3,22) respectively, is the most suitable choice as search state for the separate classifications of pattern P_2 with pattern P_2 as start state and vice-versa respectively.

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